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by

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Empirical Bayes Tests For Some Non-exponential Distribution Family

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Abstract

Empirical Bayes tests for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ in the distribution family having density $f(x|\theta) = I_{[0 \leq x \leq \theta]} a(x)/A(\theta)$ are considered under the linear loss. We construct the empirical Bayes tests δ_n and $\bar{\delta}_n$ and prove that both of them have a good asymptotic property. As a special case, taking $a(x) = 1$ leads to a result obtained by Karunamuni (1999). But our result gets rid of the condition that the critical point b_G falls in some interval $[0, \rho_0]$, where $\rho_0 < \theta_0$ is a known constant.

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§ 1 Introduction

Let X denote a random variable having density function

$$f(x|\theta) = \frac{a(x)}{A(\theta)} I_{[0 \leq x \leq \theta]}, \quad (1.1)$$

where $a(x)$ is a function on $[0, \infty)$, and continuous, positive for $x > 0$, $A(\theta) = \int_0^\theta a(x)dx < \infty$ for every $\theta \geq 0$, θ is the parameter, which is distributed according to an unknown prior distribution G on $[0, \infty)$.

We consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0 > 0$, is a known constant.

Let $a = i$ be the action in favor of H_i . For the parameter θ and action a , we use the loss function

$$l(\theta, a) = a(\theta_0 - \theta)I_{[\theta \leq \theta_0]} + (1 - a)(\theta - \theta_0)I_{[\theta > \theta_0]}. \quad (1.2)$$

Assume that

$$\int_{\Omega} \theta dG(\theta) < \infty. \quad (1.3)$$

Define

$$\alpha_G(x) = \int_x^\infty \frac{1}{A(\theta)} dG(\theta),$$

and

$$\psi_G(x) = \int_x^\infty \frac{\theta}{A(\theta)} dG(\theta).$$

By Fubini Theorem,

$$\begin{aligned} & \int_0^\infty a(x) \alpha_G(x) dx \\ &= \int_0^\infty a(x) \int_x^\infty \frac{1}{A(\theta)} dG(\theta) dx \\ &= \int_0^\infty \int_0^\infty \frac{a(x)}{A(\theta)} I_{[0 < x < \theta]} dx dG(\theta) = 1 \end{aligned} \quad (1.4)$$

and

$$\begin{aligned} & \int_0^\infty a(x) \psi_G(x) dx \\ &\leq \int_0^\infty a(x) \int_x^\infty \frac{\theta}{A(\theta)} dG(\theta) dx \\ &= \int_0^\infty \theta \int_0^\infty \frac{a(x)}{A(\theta)} I_{[0 < x < \theta]} dx dG(\theta) \\ &= \int_0^\infty \theta dG(\theta) < \infty. \end{aligned} \quad (1.5)$$

Let $W(x) = \theta_0 \alpha_G(x) - \psi_G(x)$. Using (1.4) and (1.5), we have

$$\begin{aligned} & \int_0^\infty |W(x)|a(x)dx \\ & < \int_0^\infty [\theta_0 \alpha_G(x) + \psi_G(x)]a(x)dx < \infty. \end{aligned} \quad (1.6)$$

A test $\delta(x)$ is defined to be a measurable mapping from $(0, \infty)$ into $[0, 1]$ so that $\delta(x) = P\{\text{accepting } H_1 | X = x\}$, i.e., $\delta(x)$ is the probability of accepting H_1 when $X = x$ is observed.

Let $R(G, \delta)$ denote the Bayes risk of the test δ when G is the prior distribution. Then $R(G, \delta)$ can be expressed as

$$\begin{aligned} R(G, \delta) &= C_G + \int_0^\infty \int_0^\infty \delta(x)(\theta_0 - \theta)f(x|\theta)dx dG(\theta) \\ &= C_G + \int_0^\infty \delta(x) \left[\int_x^\infty (\theta_0 - \theta) \frac{1}{A(\theta)} dG(\theta) \right] a(x) dx \\ &= C_G + \int_0^\infty \delta(x) [\theta_0 \alpha_G(x) - \psi_G(x)] a(x) dx \\ &= C_G + \int_0^\infty \delta(x) W(x) a(x) dx \\ &= C_G + \int_0^\infty \delta(x) [\theta_0 - \phi_G(x)] \alpha_G(x) a(x) dx, \end{aligned} \quad (1.7)$$

where

$$C_G = \int_0^\infty (\theta - \theta_0) I_{[\theta > \theta_0]} dG(\theta),$$

and

$$\phi_G(x) = E[\theta | X = x] = \frac{\psi_G(x)}{\alpha_G(x)}.$$

Here, $\phi_G(x)$ is the posterior mean of θ given $X = x$. $\phi_G(x)$ is continuous and increasing in x .

From (1.7), we see that a Bayes test δ_G is determined by

$$\begin{aligned} \delta_G(x) &= \begin{cases} 1 & \text{if } W(x) \leq 0 \\ 0 & \text{if } W(x) > 0 \end{cases} \\ &= \begin{cases} 1 & \text{if } \phi_G(x) \geq \theta_0 \\ 0 & \text{if } \phi_G(x) < \theta_0. \end{cases} \end{aligned} \quad (1.8)$$

The minimum Bayes risk is

$$R(G, \delta_G) = C_G + \int_0^\infty \delta_G(x) W(x) a(x) dx. \quad (1.9)$$

To exclude trivial cases, we assume that

$$\begin{cases} \phi_G(0) < \theta_0 \\ G(\theta_0) \neq 1 \end{cases} \quad (1.10)$$

From (1.10), we see that $\phi_G(x)$ is strictly increasing and there exists a unique point $b_G < \theta_0$ such that $\phi_G(b_G) = \theta_0$. $\phi_G(x) < \theta_0$ for $x < b_G$, and $\phi_G(x) > \theta_0$ for $x > b_G$. Therefore, the Bayes test δ_G can be represented as

$$\delta_G(x) = \begin{cases} 1 & \text{if } x \geq b_G \\ 0 & \text{if } x < b_G. \end{cases}$$

We assume that, for some constant $B > 0$,

$$\sup_{0 \leq x \leq \theta_0 + 1} |\alpha_G^{(i)}(x)| \leq B, \quad (1.11)$$

where $i = 0, 1, \dots, r$ and $r \geq 1$. Note that (1.11) implies that $G'(x)$ exists for $0 \leq x \leq \theta_0 + 1$. Furthermore, we assume that

$$G'(b_G) \neq 0. \quad (1.12)$$

We will deal with this testing problem via the empirical Bayes approach. The empirical Bayes approach was introduced first by Robbins (1956, 1964). Let X_1, X_2, \dots, X_n denote the observations from n independent past experiences. Let X be the present observation. Denote $\widetilde{X}_n = (X_1, X_2, \dots, X_n)$. An empirical Bayes test $\delta_n(X, \widetilde{X}_n)$ is defined to be the probability of accepting H_1 when X and \widetilde{X}_n are observed. Let $R(G, \delta_n | \widetilde{X}_n)$ denote the Bayes risk of δ_n conditioning on \widetilde{X}_n and $R(G, \delta) = E[R(G, \delta | \widetilde{X}_n)]$ the overall (unconditional) Bayes risk of δ_n .

Since $R(G, \delta_G)$ is the minimum Bayes risk, $R(G, \delta_n | \widetilde{X}_n) - R(G, \delta_G) \geq 0$ for all \widetilde{X}_n and for all n . Thus, the regret $R(G, \delta_n) - R(G, \delta_G) \geq 0$ for all n . The nonnegative regret $R(G, \delta_n) - R(G, \delta_G)$ is often used as a measure of performance of the empirical Bayes test of δ_n .

Empirical Bayes problem for the Uniform $(0, \theta)$, a special case of (1.1), was studied by a number of authors: Fox (1978), Van Houwelingen (1987), Nogami (1988), Liang (1990) and Karunamuni (1999). For the distribution family having density (1.1), Gupta and Hsiao (1983) considered the empirical Bayes rules in the selection problem formulation, Datta (1991) studied the empirical Bayes rule in the estimation problem formulation.

In this paper, we consider the empirical Bayes rule in the testing problem formulation. For clarity, we consider the different cases of $a(x)$:

Case 1 : $a(x) \rightarrow a_0$, where $0 < a_0 < \infty$, as $x \downarrow 0$,

Case 2 : $a(x) \downarrow 0$ as $x \downarrow 0$,

Case 3 : $a(x) \uparrow \infty$ as $x \downarrow 0$,

Case 4 : $a(x) \rightarrow 0$ as $x \downarrow 0$,

Case 5 : $a(x) \rightarrow \infty$ as $x \downarrow 0$,

where \uparrow (or \downarrow) stands for “goes to increasingly” (or “goes to decreasingly”, respectively).

Although Case 2 is the special case of Case 4 and Case 3 is the special case of Case 5, our approach (or result) is a little different between Case 2 and Case 4. Also our approach is different between Case 3 and Case 5. So we treat Case 2 and Case 3 as the separate cases.

Define

$$\mathcal{G}_1 = \{G : G \text{ satisfies (1.10), (1.11) and (1.12)}\}.$$

For some $\xi > 0$, $L > 0$ and $b_0 > 0$ in Cases 2, 3, 4, 5, $b_0 = 0$ in case 1, define

$$\mathcal{G}_2 = \{G : G \text{ satisfies (1.10) and (1.11), } b_G > b_0, \min_{x \in N_{b_G}(\xi)} |W'(x)| \geq L\},$$

where $N_{b_G}(\xi) = \{x : 0 \vee (b_G - \xi) \leq x \leq (b_G + \xi) \wedge \theta_0\}$. Then we can construct a Bayes test δ_n such that its regret has a convergence rate of order $O(n^{-\frac{2r}{2r+1}})$ or $O(n^{-\frac{2r}{2r+1}} \epsilon_n^{-1})$ (different rates in different cases) for any $G \in \mathcal{G}_1$, where ϵ_n is any prespecified (large) positive sequence such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. And we can also construct another empirical Bayes test $\bar{\delta}_n$ such that its regret has a uniform convergence rate of order $O(n^{-\frac{2r}{2r+1}})$ over the class \mathcal{G}_2 . Taking $a(x) = 1$, we would get that $\sup_{G \in \mathcal{G}_2} R(G, \delta) - R(G, \bar{\delta}) = O(n^{-\frac{2r}{2r+1}})$, which is a result obtained by Karunamuni (1999). But we exclude the condition that b_G falls in some known interval $[0, \rho_0]$, where $\rho_0 < \theta_0$ is a known constant.

The paper is organized as follows: §1 gives the introduction; §2 constructs the empirical Bayes test δ_n ; §3 proves that the empirical Bayes test has a good asymptotic property. §4 proves the lemmas stated in §3.

§ 2 Construction of Empirical Bayes Tests

We use the kernel method to construct the empirical Bayes tests. Let $K_0(y)$ be a Borel-measurable, bounded function vanishing outside the interval $[0, 1]$ such that

$$\int_0^1 y^j K_0(y) dy = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j = 1, 2, \dots, r-1, \end{cases} \quad (2.1)$$

and let

$$K_1(y) = \int_0^y K_0(s) ds. \quad (2.2)$$

We may let B_1 be a positive constant such that $|K_0(y)| \leq B_1$ for all $y \in [0, 1]$. Let ϵ_n be any (large) positive sequence such that $\epsilon_n \rightarrow 0$. Without loss of generality, assume

$\epsilon_n > \frac{1}{\log n}$. Let u_n be a positive sequence such that $u_n \leq 1$ and $u_n \rightarrow 0$ as $n \rightarrow \infty$. For any $x \in (0, \infty)$, define, for Case 1, Case 2 and Case 4,

$$\alpha_n(x) = \frac{1}{nu} \sum_{j=1}^n \frac{K_0\left(\frac{X_j - x}{u}\right)}{a(X_j)}, \quad (2.3.1)$$

$$\psi_n(x) = \frac{x}{nu} \sum_{j=1}^n \frac{K_0\left(\frac{X_j - x}{u}\right)}{a(X_j)} - \frac{1}{n} \sum_{j=1}^n \frac{K_1\left(\frac{X_j - x}{u}\right)}{a(X_j)}. \quad (2.3.2)$$

and for Case 3 and Case 5,

$$\alpha_n(x) = -\frac{1}{nu} \sum_{j=1}^n \frac{K_0\left(\frac{x - X_j}{u}\right)}{a(X_j)}, \quad (2.4.1)$$

$$\psi_n(x) = -\frac{x}{nu} \sum_{j=1}^n \frac{K_0\left(\frac{x - X_j}{u}\right)}{a(X_j)} + \frac{1}{n} \sum_{j=1}^n \frac{K_1\left(\frac{x - X_j}{u}\right)}{a(X_j)}. \quad (2.4.2)$$

Let $W_n(x) = \theta_0 \alpha_n(x) - \psi_n(x)$. We shall show later that $W_n(x)$ is an asymptotically unbiased and consistent estimators of $W(x)$ (Lemma 3.1). For any $G \in \mathcal{G}_1$, we propose an empirical Bayes test $\delta_n(x, \widetilde{X}_n)$ by

$$\delta_n = \begin{cases} 1 & \text{if } (x > \theta_0) \text{ or } (d_n \leq x \leq \theta_0 \text{ and } W_n(x) \leq 0), \\ 0 & \text{if } (x < d_n) \text{ or } (d_n \leq x \leq \theta_0 \text{ and } W_n(x) > 0), \end{cases} \quad (2.5)$$

where

$$d_n = \begin{cases} 0 & \text{for Case 1,} \\ \max\{x : a(x) < \eta_n\} & \text{for Case 2 and Case 4,} \\ \max\{x : a(x) > \frac{1}{\eta_n}, a(x) > \frac{1}{u}\} + u & \text{for Case 3 and Case 5,} \end{cases} \quad (2.6)$$

and $\eta_n = \epsilon_n^{\frac{2r+1}{2r}}$. Since $\eta_n \rightarrow 0$ as $n \rightarrow \infty$, we know that $d_n \rightarrow 0$ as $n \rightarrow \infty$. Suppose N_0 is the smallest positive integer such that $b_G > d_{N_0}$. Then $b_G > d_n$ for $n > N_0$. When we discuss δ_n , we always assume that $n > N_0$ without further mention in this paper.

Note that $W(x) \leq 0$ if $x \in (0, b_G]$; $W(x) \geq 0$ if $x \in [b_G, \theta_0]$. Then the conditional regret of δ_n can be expressed as

$$\begin{aligned} R(G, \delta_n | \widetilde{X}_n) - R(G, \delta) &= \int_0^\infty (\delta_n - \delta) W(x) a(x) dx \\ &= \int_{d_n}^{b_G} I_{[W_n(x) \leq 0]} W(x) a(x) dx \\ &\quad + \int_{b_G}^{\theta_0} I_{[W_n(x) > 0]} |W(x)| a(x) dx \end{aligned} \quad (2.7)$$

and the unconditional regret of δ_n becomes

$$\begin{aligned} R(G, \delta_n) - R(G, \delta) &= \int_{d_n}^{b_G} P(W_n(x) \leq 0) W(x) a(x) dx \\ &\quad + \int_{b_G}^{\theta_0} P(W_n(x) > 0) |W(x)| a(x) dx. \end{aligned} \quad (2.8)$$

For $G \in \mathcal{G}_2$, we also propose another empirical Bayes procedure $\bar{\delta}_n$ by

$$\delta_n = \begin{cases} 1 & \text{if } (x > \theta_0) \text{ or } (b_0 \leq x \leq \theta_0 \text{ and } W_n(x) \leq 0), \\ 0 & \text{if } (x < b_0) \text{ or } (b_0 \leq x \leq \theta_0 \text{ and } W_n(x) > 0). \end{cases} \quad (2.9)$$

Then the unconditional regret of $\bar{\delta}_n$ becomes

$$\begin{aligned} R(G, \bar{\delta}_n) - R(G, \delta) &= \int_{b_0}^{b_G} P(W_n(x) \leq 0) W(x) a(x) dx \\ &\quad + \int_{b_G}^{\theta_0} P(W_n(x) > 0) |W(x)| a(x) dx. \end{aligned} \quad (2.10)$$

In the following section, the convergence rate of unconditional regret of δ_n for every $G \in \mathcal{G}_1$ and the uniform convergence rate of unconditional regret of $\bar{\delta}_n$ over \mathcal{G}_2 are considered.

§3 Asymptotic Optimality of δ_n and $\bar{\delta}_n$

The convergence rates of δ_n and $\bar{\delta}_n$ depend on the properties of $W(x)$ and $W_n(x)$. The more information about $W(x)$ and $W_n(x)$ (including $a(x)$) is used, the more accurate rate we will get. So firstly, we dig out a few properties of $W_n(x)$ and $W(x)$. That is a few lemmas, whose proofs are left to §4. Then we state a well-known fact. Following that, two theorems about asymptotic optimalities of δ_n and $\bar{\delta}_n$ are given.

Note that

$$W_n(x) = \theta_0 \alpha_n(x) - \psi_n(x) = \frac{1}{n} \sum_{j=1}^n V(X_j, x, n), \quad (3.1)$$

where, for Case 1, Case 2 and Case 4,

$$V(X_j, x, n) = \frac{\theta_0 - x}{u} \times \frac{K_0(\frac{X_j - x}{u})}{a(X_j)} - \frac{K_1(\frac{X_j - x}{u})}{a(X_j)}, \quad (3.2)$$

and for Case 3 and Case 5,

$$V(X_j, x, n) = -\frac{\theta_0 - x}{u} \times \frac{K_0(\frac{x - X_j}{u})}{a(X_j)} + \frac{K_1(\frac{x - X_j}{u})}{a(X_j)}. \quad (3.3)$$

Let $\bar{W}(x, n) = E[V(X_j, x, n)]$ and $Z_{jn} = V(X_j, x, n) - \bar{W}(x, n)$. Then we have

Lemma 3.1 $\bar{W}(x, n)$ can be expressed as

$$\bar{W}(x, n) = W(x) + u^r W(x, n), \quad (3.4)$$

where $W(x, n)$ is some function such that for all $x \in [d_n \wedge b_0, \theta_0]$,

$$|W(x, n)| \leq \frac{2\theta_0(r+2)+1}{(r+2)!} B B_1 \equiv B_2. \quad (3.5)$$

Lemma 3.2 For $x \in [0, \theta_0]$, $|W(x)| \leq 2\theta_0 B$.

Lemma 3.3 For any fixed n , Z_{jn} are i.i.d., and for $x \in [d_n \wedge b_0, \theta_0]$, $E Z_{jn} = 0$,

$$\sigma_n^2 \equiv E Z_{jn}^2 \leq \frac{B_3^2}{u a^*(x)}, \quad (3.6)$$

and

$$\frac{E|Z_{jn}|^3}{\sigma_n^2} \leq 2\theta_0 B + B_2 + \frac{2(\theta_0 B_1 + 1)}{u a^*(x)}, \quad (3.7)$$

where $B_3 = (2\theta_0 + 1)B_1\sqrt{B}$, and

$$a^*(x) = \begin{cases} \min\{a(s) : x \leq s \leq x + u\} & \text{for Cases 1, 2, 4,} \\ \min\{a(s) : x - u \leq s \leq x\} & \text{for Cases 3, 5.} \end{cases} \quad (3.8)$$

From Lemma 3.3, we see that

$$P(W_n(x) > 0) = P\left(\frac{1}{\sqrt{n E Z_{jn}^2}} \sum_{j=1}^n Z_{jn} > -\sqrt{n \sigma_n^{-2}} \bar{W}(x, n)\right), \quad (3.9)$$

and

$$P(W_n(x) \leq 0) = P\left(\frac{1}{\sqrt{n E Z_{jn}^2}} \sum_{j=1}^n Z_{jn} \leq -\sqrt{n \sigma_n^{-2}} \bar{W}(x, n)\right). \quad (3.10)$$

Based on Lemma 3.1, we obtain the following useful result:

Lemma 3.4 For $x \in [d_n \wedge b_0, \theta_0]$,

$$W(x) > 2B_2 u^r \implies \bar{W}(x, n) \geq 0 \text{ and } \frac{W(x)}{\bar{W}(x, n)} \leq 2, \quad (3.11)$$

and

$$W(x) < -2B_2u^r \implies \overline{W}(x, n) \leq 0 \text{ and } \left| \frac{W(x)}{\overline{W}(x, n)} \right| \leq 2. \quad (3.12)$$

Lemma 3.4 allows us to replace $\overline{W}(x, n)$ with $W(x)$ in (3.9) and (3.10). That makes things a little easier since we will see that $W(x)$ does not depend on n and has a few good properties.

The above four hold for any $G \in \mathcal{G}_1$ or \mathcal{G}_2 . Note the bounds in above lemmas do not depend on G . So they are the uniform bounds over \mathcal{G}_1 and \mathcal{G}_2 . Next two lemmas give the results related to some $G \in \mathcal{G}_1$, which will be used only when we consider δ_n .

Lemma 3.5 *For any $G \in \mathcal{G}_1$, there exist c_{G1} and c_{G2} such that $0 < c_{G1} < b_G < c_{G2} < \theta_0$, and for all $x \in [c_{G1}, c_{G2}]$,*

$$|W'(x)| \geq \frac{G'(b_G)}{2A(b_G)}(\theta_0 - b_G) \equiv B_{G1}^{-1}, \quad (3.13)$$

for all $x \in [0, c_{G1}] \cup [c_{G2}, \theta_0]$,

$$|W(x)| \geq W(c_{G1}) \wedge |W(c_{G2})| \equiv B_{G2}. \quad (3.14)$$

Lemma 3.6 *For any $G \in \mathcal{G}_1$, let $U_G = \max\{a(x) : c_{G1} \leq x \leq c_{G2}\}$. Then there exists an integer $N_1(> N_0)$ such that for $n > N_1$,*

$$\int_{d_n}^{\theta_0} I_{[|W(x)| \leq 2B_2u^r]} a(x) dx \leq 4B_2B_{G1}U_Gu^r. \quad (3.15)$$

Let $L_G = \min\{a(x) : \frac{c_{G1}}{2} \leq x \leq c_{G2} + \frac{\theta_0 - c_{G2}}{2}\}$. Then there exists an integer $N_2(> N_1)$ such that for $n > N_1$,

$$a^*(x) \begin{cases} \geq L_G & \text{for } x \in [c_{G1}, c_{G2}], \\ > \eta_n & \text{for } x \in [d_n, d_n \vee c_{G1}] \cup [c_{G2}, \theta_0]. \end{cases} \quad (3.16)$$

When we consider the uniform rate of convergence of $[R(G, \bar{\delta}_n) - R(G, \delta)]$, we need the following two lemmas.

Lemma 3.7 *For all $G \in \mathcal{G}_2$ and all $x \in [0 \vee (b_G - \xi), (b_G + \xi) \wedge \theta_0]$,*

$$|W'(x)| \geq L, \quad (3.17)$$

for all $G \in \mathcal{G}_2$ and all $x \in [0, 0 \vee (b_G - \xi)] \cup [(b_G + \xi) \wedge \theta_0, \theta_0]$

$$|W(x)| \geq L\xi. \quad (3.18)$$

Lemma 3.8 *Let $U_a = \max\{a(x) : b_0 \leq x \leq \theta_0\}$. Then there exists an integer \bar{N}_1 such that for all $n > \bar{N}_1$,*

$$\int_{b_0}^{\theta_0} I_{[|W(x)| \leq 2B_2 u^r]} a(x) dx \leq \frac{4B_2 U_a}{L} u^r. \quad (3.19)$$

Let $L_a = \min\{a(x) : \frac{b_0}{2} \leq x \leq \theta_0 + 1\}$. Then there exists an integer $\bar{N}_2 > \bar{N}_1$ such that for all $n > \bar{N}_2$ and $x \in [b_0, \theta_0]$

$$a^*(x) \geq L_a > 0. \quad (3.20)$$

Next we state a general well-known result. It is about the non-uniform estimate of the distance between the distribution of a sum of i.i.d. random variables and the normal distribution.

Result *Let X_1, X_2, \dots, X_n be i.i.d random variables, $EX_1 = 0$, $EX_1^2 = \sigma^2 > 0$, $E|X_1|^3 < \infty$. Then for all x*

$$|F_n(x) - \Psi(x)| \leq A \frac{\rho}{\sqrt{n}(1 + |x|)^3}.$$

Here $\Psi(x)$ is the c.d.f. of $N(0, 1)$, $F_n(x)$ and ρ are given by

$$F_n(x) = P\left(\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^n X_j \leq x\right), \quad \rho = \frac{E|X_1|^3}{\sigma^3}.$$

Remark *The above result can be found in Petrov (1975, pp125 Theorem 14) or Michel (1981). Here A is independent of n . Michel proved $A < 30.54$.*

From the above result, we see that, for any fixed n , if $x > 0$,

$$\begin{aligned} & P\left(\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^n X_j > x\right) \quad (3.21) \\ &= 1 - F_n(x) \\ &\leq 1 - \Psi(x) + A \frac{\rho}{\sqrt{n}(1 + |x|)^3} \\ &\leq 1 - \Psi(x) + A \frac{E|X_1|^3}{\sqrt{n}|x|\sigma^3}, \end{aligned}$$

if $x < 0$,

$$\begin{aligned}
& P\left(\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^n X_j \leq x\right) \quad (3.22) \\
&= F_n(x) \\
&\leq \Psi(x) + A \frac{\rho}{\sqrt{n}(1+|x|)^3} \\
&\leq \Psi(x) + A \frac{E|X_1|^3}{\sqrt{n}|x|\sigma^3}.
\end{aligned}$$

Now, we prove our main results. The first one is related to δ_n :

Theorem 3.9 *Let $u = u(n) = n^{-\frac{1}{2r+1}}$ for Cases 1, 2, 3, 5 and $n^{-\frac{1}{2r+1}}\epsilon_n^{-\frac{1}{2r}}$ for Case 4. Then we have, for every $G \in \mathcal{G}_1$, as $n \rightarrow \infty$, for Case 1, Case 2, Case 3, Case 5,*

$$\lim_{n \rightarrow \infty} n^{\frac{2r}{2r+1}} [R(G, \delta_n) - R(G, \delta)] < 16B_2^2 B_{G1} U_G + 2B_G < \infty, \quad (3.23.1)$$

and for Case 4,

$$\lim_{n \rightarrow \infty} n^{\frac{2r}{2r+1}} \epsilon_n [R(G, \delta_n) - R(G, \delta)] < 16B_2^2 B_{G1} U_G + 2B_G < \infty, \quad (3.23.2)$$

where

$$B_G = \begin{cases} 2\theta_0 B A(\theta_0) + \frac{2B_{G1} B_3^2 U_G}{L_G} + 2A(2\theta_0 B + B_2 + \frac{2\theta_0 B_1 + 1}{a_{min}})A(\theta_0) & \text{for Cases 1, 5,} \\ 2\theta_0 B A(\theta_0) + \frac{2B_{G1} B_3^2 U_G}{L_G} + 2A[(2\theta_0 B + B_2)A(\theta_0) + (2\theta_0 B_1 + 1)\theta_0] & \text{for Cases 2, 3,} \\ 2\theta_0 B A(\theta_0) + \frac{2B_{G1} B_3^2 U_G}{L_G} + 2A(2\theta_0 B + B_2 + 2\theta_0 B_1 + 1)A(\theta_0) & \text{for Case 4,} \end{cases}$$

and $a_{min} = \min\{a(x) : 0 \leq x \leq \theta_0\} > 0$ in Cases 1, 5.

Proof. From (2.8),

$$\begin{aligned}
R(G, \delta_n) - R(G, \delta) &= \int_{d_n}^{b_G} P_n(W_n(x) \leq 0) W(x) a(x) I_{[0 < W(x) \leq 2B_2 u^r]} dx \quad (3.24) \\
&+ \int_{b_G}^{\theta_0} P_n(W_n(x) > 0) |W(x)| a(x) I_{[-2B_2 u^r \leq W(x) < 0]} dx \\
&+ \int_{d_n}^{b_G} P_n(W_n(x) \leq 0) W(x) a(x) I_{[W(x) > 2B_2 u^r]} dx \\
&+ \int_{b_G}^{\theta_0} P_n(W_n(x) > 0) |W(x)| a(x) I_{[W(x) < -2B_2 u^r]} dx \\
&\equiv I + II + III + IV.
\end{aligned}$$

Part I and Part II are easy to handle, since we have Lemma 3.6. Using (3.15), we have, as $n > N_1$,

$$I \leq 2B_2 u^r \int_{d_n}^{b_G} I_{[0 < W(x) \leq 2B_2 u^r]} a(x) dx \leq 8B_2^2 B_{G1} U_G u^{2r}, \quad (3.25)$$

and

$$II \leq 2B_2 u^r \int_{b_G}^{\theta_0} I_{[-2B_2 u^r \leq W(x) < 0]} a(x) dx \leq 8B_2^2 B_{G1} U_G u^{2r}. \quad (3.26)$$

Part III and Part IV are a little more complicated. We treat Part III first. Using (3.9), (3.11) and (3.22), we have

$$\begin{aligned} III &\leq \int_{d_n}^{b_G} P\left(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^n Z_{jn} \leq -\sqrt{n\sigma_n^{-2}} \bar{W}(x, n)\right) I_{[W(x) > 2B_2 u^r]} W(x) a(x) dx \\ &\leq \int_{d_n}^{b_G} P\left(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^n Z_{jn} \leq -\frac{1}{2}\sqrt{n\sigma_n^{-2}} W(x)\right) I_{[W(x) > 2B_2 u^r]} W(x) a(x) dx \\ &\leq \int_{d_n}^{b_G} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}} W(x)\right) W(x) a(x) dx + \int_{d_n}^{b_G} \frac{AE|Z_{jn}|^3 W(x) a(x) dx}{\sqrt{n} \times \left| -\frac{1}{2}\sqrt{n\sigma_n^{-2}} W(x) \right| \times \sigma_n^3} \\ &\leq \int_{d_n}^{c_{G1} \vee d_n} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}} W(x)\right) W(x) a(x) dx \\ &\quad + \int_{c_{G1} \vee d_n}^{b_G} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}} W(x)\right) W(x) a(x) dx \\ &\quad + \int_{d_n}^{b_G} \frac{2A}{n} \times \frac{E|Z_{jn}|^3}{\sigma_n^2} a(x) dx \\ &\equiv V + VI + VII. \end{aligned} \quad (3.27)$$

Using (3.6), (3.16) and (3.14), we have

$$\begin{aligned} V &\leq \int_{d_n}^{c_{G1} \vee d_n} \Phi\left(-\frac{1}{2}\sqrt{n} \frac{\sqrt{u\eta_n}}{B_3} B_{G2}\right) W(x) a(x) dx \\ &= \Phi\left(-\frac{B_{G2}}{2B_3} \sqrt{nu\eta_n}\right) \int_{d_n}^{c_{G1} \vee d_n} W(x) a(x) dx \\ &\leq \Phi\left(-\frac{B_{G2}}{2B_3} \sqrt{nu\eta_n}\right) \int_0^{b_G} W(x) a(x) dx \\ &\leq \Phi\left(-\frac{B_{G2}}{2B_3} \sqrt{nu\eta_n}\right) \times 2\theta_0 B A(\theta_0). \end{aligned}$$

Since $\sqrt{nu\eta_n} > \log n$, there exists an integer $N_3 (> N_2)$ such that for $n > N_3$, $\Phi\left(-\frac{B_{G2}}{2B_3} \sqrt{nu\eta_n}\right) \leq \frac{1}{n}$ and

$$V \leq \frac{2\theta_0 B A(\theta_0)}{n} \leq \frac{2\theta_0 B A(\theta_0)}{nu}. \quad (3.28)$$

As for Part VI, using (3.6) and (3.16), and making change of variable $y = \frac{1}{2B_3} \sqrt{nuL_G} W(x)$, we obtain

$$VI \leq \int_{c_{G1} \vee d_n}^{b_G} \Phi\left(-\frac{1}{2B_3} \sqrt{nu a^*(x)} W(x)\right) W(x) a(x) dx \quad (3.29)$$

$$\begin{aligned}
&\leq \int_{c_{G1} \vee d_n}^{b_G} U_G \Phi\left(-\frac{1}{2B_3} \sqrt{nuL_G} W(x)\right) W(x) dx \\
&\leq -B_{G1} U_G \int_{c_{G1} \vee d_n}^{b_G} \Phi\left(-\frac{1}{2B_3} \sqrt{nuL_G} W(x)\right) W(x) W'(x) dx \\
&\leq \frac{4B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu} \int_0^{\frac{1}{2B_3} \sqrt{nuL_G} W(c_{G1} \vee d_n)} \Phi(-y) y dy \\
&\leq \frac{4B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu} \int_0^\infty \Phi(-y) y dy \\
&\leq \frac{2B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu}.
\end{aligned}$$

We consider VII in different cases. For Case 1 and Case 5,

$$\begin{aligned}
VII &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{d_n}^{b_G} \frac{2\theta_0 B_1 + 1}{ua^*(x)} a(x) dx \quad (3.30.1) \\
&\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)A(\theta_0)}{a_{min} nu}.
\end{aligned}$$

For Case 2 and Case 3,

$$\begin{aligned}
VII &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{d_n}^{b_G} \frac{2\theta_0 B_1 + 1}{ua(x)} a(x) dx \quad (3.30.2) \\
&\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 + 1)B_1 \theta_0}{nu}.
\end{aligned}$$

For Case 4,

$$\begin{aligned}
VII &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{d_n}^{b_G} \frac{2\theta_0 B_1 + 1}{u\eta_n} a(x) dx \quad (3.30.3) \\
&\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + 2A(2\theta_0 B_1 + 1) \times A(\theta_0) \times \frac{1}{nu\eta_n}.
\end{aligned}$$

Combining (3.28), (3.29) and (3.30), we get that when $n \geq N_3$,

$$III \leq \begin{cases} B_G \times \frac{1}{nu} & \text{for Cases 1, 2, 3, 5,} \\ B_G \times \frac{1}{nu\eta_n} & \text{for Case 4.} \end{cases} \quad (3.31)$$

Now we deal with IV. Similar to III, we get

$$\begin{aligned}
IV &\leq \int_{b_G}^{\theta_0} P\left(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^n Z_{jn} \geq -\sqrt{n\sigma_n^{-2}} \bar{W}(x, n)\right) I_{[W(x) < -2B_2 u^r]} |W(x)| a(x) dx \quad (3.32) \\
&\leq \int_{b_G}^{\theta_0} P\left(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^n Z_{jn} \geq -\frac{1}{2} \sqrt{n\sigma_n^{-2}} W(x)\right) I_{[W(x) < -2B_2 u^r]} |W(x)| a(x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{b_G}^{\theta_0} [1 - \Psi(\frac{1}{2}\sqrt{n\sigma_n^{-2}}|W(x)|)]|W(x)|a(x)dx + \int_{b_G}^{\theta_0} \frac{AE|Z_{jn}|^3|W(x)|a(x)dx}{\sqrt{n} \times \frac{1}{2}\sqrt{n\sigma_n^{-2}}|W(x)| \times \sigma_n^3} \\
&\leq \int_{c_{G2}}^{\theta_0} [1 - \Psi(\frac{1}{2}\sqrt{n\sigma_n^{-2}}|W(x)|)]|W(x)|a(x)dx \\
&\quad + \int_{b_G}^{c_{G2}} [1 - \Psi(-\frac{1}{2}\sqrt{n\sigma_n^{-2}}|W(x)|)]|W(x)|a(x)dx \\
&\quad + \int_{b_G}^{\theta_0} \frac{2A}{n} \times \frac{E|Z_{jn}|^3}{\sigma_n^2} a(x)dx \\
&\equiv VIII + IX + X.
\end{aligned}$$

Using (3.6), (3.16) and (3.14), we have

$$\begin{aligned}
VIII &\leq \int_{c_{G2}}^{\theta_0} [1 - \Phi(\frac{1}{2}\sqrt{n}\frac{\sqrt{u\eta_n}}{B_3}B_{G2})]|W(x)|a(x)dx \\
&= [1 - \Phi(\frac{B_{G2}}{2B_3}\sqrt{nu\eta_n})] \int_{c_{G2}}^{\theta_0} |W(x)|a(x)dx \\
&\leq [1 - \Phi(\frac{B_{G2}}{2B_3}\sqrt{nu\eta_n})] \int_{b_G}^{\theta_0} |W(x)|a(x)dx \\
&\leq [1 - \Phi(\frac{B_{G2}}{2B_3}\sqrt{nu\eta_n})] \times 2\theta_0 BA(\theta_0).
\end{aligned}$$

By the symmetric property of $\Psi(x)$, for $n > N_3$, $1 - \Phi(\frac{B_{G2}}{2B_3}\sqrt{nu\eta_n}) \leq \frac{1}{n}$ and

$$VIII \leq \frac{2\theta_0 BA(\theta_0)}{n} \leq \frac{B_0}{nu}. \quad (3.33)$$

Similar to VII, we obtain

$$\begin{aligned}
IX &\leq \int_{b_G}^{c_{G2}} [1 - \Phi(-\frac{1}{2B_3}\sqrt{nu}a^*(x)W(x))]|W(x)|a(x)dx \quad (3.34) \\
&\leq \int_{b_G}^{c_{G2}} U_G [1 - \Phi(-\frac{1}{2B_3}\sqrt{nu}L_G W(x))]|W(x)|dx \\
&\leq -B_{G1}U_G \int_{b_G}^{c_{G2}} [1 - \Phi(-\frac{1}{2B_3}\sqrt{nu}L_G W(x))]|W(x)|W'(x)dx \\
&\leq \frac{4B_{G1}B_3^2U_G}{L_G} \times \frac{1}{nu} \int_0^{\frac{1}{2B_3}\sqrt{nu}L_G|W(c_{G2})|} [1 - \Phi(y)]ydy \\
&\leq \frac{4B_{G1}B_3^2U_G}{L_G} \times \frac{1}{nu} \int_0^\infty [1 - \Phi(y)]ydy \\
&\leq \frac{2B_{G1}B_3^2U_G}{L_G} \times \frac{1}{nu}.
\end{aligned}$$

We consider Part VII in different cases. Obviously, for Case 1 and Case 5,

$$X \leq \frac{2A(2\theta_0B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{b_G}^{\theta_0} \frac{2\theta_0B_1 + 1}{ua^*(x)} a(x)dx \quad (3.35.1)$$

$$\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)A(\theta_0)}{a_{\min} nu},$$

where $a_{\min} = \min\{a(x) : 0 \leq x \leq \theta_0\}$. For Case 2 and Case 3,

$$\begin{aligned} X &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{b_G}^{\theta_0} \frac{2\theta_0 B_1 + 1}{ua(x)} a(x) dx \\ &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)\theta_0}{nu}. \end{aligned} \quad (3.35.2)$$

For Case 4,

$$\begin{aligned} X &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{b_G}^{\theta_0} \frac{2\theta_0 B_1 + 1}{u\eta_n} a(x) dx \\ &\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + 2A(2\theta_0 B_1 + 1) \times A(\theta_0) \times \frac{1}{nu\eta_n}. \end{aligned} \quad (3.35.3)$$

Combining (3.33), (3.34) and (3.35), we get that when $n \geq N_3$,

$$IV \leq \begin{cases} B_G \times \frac{1}{nu} & \text{for Cases 1, 2, 3, 5,} \\ B_G \times \frac{1}{nu\eta_n} & \text{for Case 4.} \end{cases} \quad (3.36)$$

From (3.25), (3.26), (3.31) and (3.36), we get, for $n > N_3$,

$$R(G, \delta_n) - R(G, \delta) \leq \begin{cases} 16B_2^2 B_{G1} U_G \times u^{2r} + 2B_G \times \frac{1}{nu} & \text{for Cases 1, 2, 3, 5,} \\ 16B_2^2 B_{G1} U_G \times u^{2r} + 2B_G \times \frac{1}{nu\eta_n} & \text{for Case 4,} \end{cases} \quad (3.37)$$

For Cases 1, 2, 3, 5, $u = n^{-\frac{1}{2r+1}}$. So $u^{2r} = \frac{1}{nu} = n^{-\frac{2r}{2r+1}}$, and

$$\lim_{n \rightarrow \infty} n^{\frac{2r}{2r+1}} [R(G, \delta_n) - R(G, \delta)] \leq 16B_2^2 B_{G1} U_G + 2B_G < \infty.$$

For Case 4, we have $u = n^{-\frac{1}{2r+1}} \epsilon_n^{-\frac{1}{2r}}$, $\eta_n = \epsilon_n^{\frac{2r+1}{2r}}$. So $u^{2r} = \frac{1}{nu\eta_n} = n^{-\frac{2r}{2r+1}} \epsilon_n^{-1}$, and

$$\lim_{n \rightarrow \infty} n^{\frac{2r}{2r+1}} \epsilon_n [R(G, \delta_n) - R(G, \delta)] \leq 16B_2^2 B_{G1} U_G + 2B_G < \infty.$$

The proof is completed.

Next, we consider the uniform rate of convergence of $R(G, \bar{\delta}_n) - R(G, \delta)$.

Theorem 3.10 Take $u_n = n^{-\frac{1}{2r+1}}$. Then we have, as $n \rightarrow \infty$, for Case 1, Case 2, Case 3, Case 4, Case 5,

$$\lim_{n \rightarrow \infty} n^{\frac{2r}{2r+1}} \sup_{G \in \mathcal{G}_2} [R(G, \bar{\delta}_n) - R(G, \delta)] < \frac{4B_2^2 U_a}{L} + 2B_a < \infty. \quad (3.38)$$

Proof. In this proof, let $c_{G1} = 0 \vee (b_G - \xi)$, $c_{G2} = (b_G + \xi) \wedge \theta_0$. For any $G \in \mathcal{G}$, like (3.24), we have

$$\begin{aligned}
R(G, \bar{\delta}_n) - R(G, \delta) &= \int_{b_0}^{b_G} P_n(W_n(x) \leq 0) W(x) a(x) I_{[0 < W(x) \leq 2B_2 u^r]} dx \\
&\quad + \int_{b_G}^{\theta_0} P_n(W_n(x) > 0) |W(x)| a(x) I_{[-2B_2 u^r \leq W(x) < 0]} dx \\
&\quad + \int_{b_0}^{b_G} P_n(W_n(x) \leq 0) W(x) a(x) I_{[W(x) > 2B_2 u^r]} dx \\
&\quad + \int_{b_G}^{\theta_0} P_n(W_n(x) > 0) |W(x)| a(x) I_{[W(x) < -2B_2 u^r]} dx \\
&\equiv I + II + III + IV.
\end{aligned} \tag{3.39}$$

Using (3.19), we have, as $n > \bar{N}_1$,

$$I \leq 2B_2 u^r \int_{b_0}^{b_G} I_{[0 < W(x) \leq 2B_2 u^r]} a(x) dx \leq \frac{8B_2^2 U_a}{L} u^{2r}, \tag{3.40}$$

and

$$II \leq 2B_2 u^r \int_{b_G}^{\theta_0} I_{[-2B_2 u^r \leq W(x) < 0]} a(x) dx \leq \frac{8B_2^2 U_a}{L} u^{2r}. \tag{3.41}$$

Similar to (3.27)

$$\begin{aligned}
III &\leq \int_{b_0}^{c_{G1} \vee b_0} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}}W(x)\right) W(x) a(x) dx \\
&\quad + \int_{c_{G1} \vee b_0}^{b_G} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}}W(x)\right) W(x) a(x) dx \\
&\quad + \int_{b_0}^{b_G} \frac{2A}{n} \times \frac{E|Z_{jn}|^3}{\sigma_n^2} a(x) dx \\
&\equiv V + VI + VII.
\end{aligned}$$

Consider Part V first. If $b_0 < c_{G1}$,

$$\begin{aligned}
V &\leq \int_{b_0}^{c_{G1} \vee b_0} \Phi\left(-\frac{1}{2}\sqrt{n}\frac{\sqrt{uL_a}}{B_3}L\xi\right) W(x) a(x) dx \\
&= \Phi\left(-\frac{L\xi}{2B_3}\sqrt{nuL_a}\right) \int_{b_0}^{c_{G1} \vee b_0} W(x) a(x) dx \\
&\leq \Phi\left(-\frac{L\xi}{2B_3}\sqrt{nuL_a}\right) \int_0^{\theta_0} W(x) a(x) dx \\
&\leq \Phi\left(-\frac{L\xi}{2B_3}\sqrt{nuL_a}\right) 2\theta_0 B A(\theta_0).
\end{aligned}$$

Since $\sqrt{nuL_a} > \log n$, there exists an integer $\bar{N}_3 (> \bar{N}_2)$ such that for $n > \bar{N}_3$, $\Phi(-\frac{L\xi}{2B_3}\sqrt{nuL_a}) \leq \frac{1}{n}$ and

$$V \leq \frac{2\theta_0 B}{n} A(\theta_0). \quad (3.42)$$

Now consider VI.

$$\begin{aligned} VI &\leq \int_{c_{G1} \vee b_0}^{b_G} \Phi\left(-\frac{1}{2B_3} \sqrt{nu a^*(x)} W(x)\right) W(x) a(x) dx \\ &\leq U_a \int_{c_{G1} \vee b_0}^{b_G} \Phi\left(-\frac{1}{2B_3} \sqrt{nu L_a} W(x)\right) W(x) dx \\ &\leq -L^{-1} U_a \int_{c_{G1} \vee b_0}^{b_G} \Phi\left(-\frac{1}{2B_3} \sqrt{nu L_a} W(x)\right) W(x) W'(x) dx \\ &\leq \frac{B_3^2 U_a}{LL_a} \times \frac{1}{nu} \int_0^{\frac{1}{2B_3} \sqrt{nu L_a} W(c_{G1} \vee b_0)} \Phi(-y) y dy \\ &\leq \frac{4B_3^2 U_a}{LL_a} \times \frac{1}{nu} \int_0^\infty \Phi(-y) y dy \\ &\leq \frac{2B_3^2 U_a}{LL_a} \times \frac{1}{nu}. \end{aligned} \quad (3.43)$$

For VII, using (3.7) and (3.20), we have

$$VII \leq \frac{2A}{n} (2\theta_0 B + B_2) A(\theta_0) \times \frac{1}{n} + \frac{4A\theta_0 B_1 A(\theta_0)}{L_a} \times \frac{1}{nu}. \quad (3.44)$$

Combining (3.42), (3.43) and (3.44), we get that when $n \geq \bar{N}_3$,

$$III \leq B_a \times \frac{1}{nu},$$

where

$$B_a = 2\theta_0 B A(\theta_0) + \frac{2B_3^2 U_a}{LL_a} + 2A(2\theta_0 B + B_2) A(\theta_0) + \frac{4A\theta_0 B_1 A(\theta_0)}{L_a}.$$

Similarly, we have

$$IV \leq B_a \times \frac{1}{nu}. \quad (3.45)$$

From (3.40), (3.41), (3.44) and (3.45), we get, for $n > \bar{N}_3$,

$$R(G, \delta_n) - R(G, \delta) \leq \frac{16B_2^2 U_a}{L} \times u^{2r} + 2B_a \times \frac{1}{nu}$$

Since $u = n^{-\frac{1}{2r+1}}$, $u^{2r} = \frac{1}{nu} = n^{-\frac{2r}{2r+1}}$. When $n > \bar{N}_3$,

$$n^{\frac{2r}{2r+1}} \sup_{G \in \mathcal{G}_2} [R(G, \delta_n) - R(G, \delta)] \leq \frac{16B_2^2 U_a}{L} + 2B_a.$$

This completes the proof.

Remark For Case 1, $b_0 = 0$. Since (1.10) implies $b_G > b_0 = 0$, we have

$$\mathcal{G}_2 = \{G : G \text{ satisfies (1.10) and (1.11), } \min_{x \in N_{b_G}(\xi)} |W(x)| \geq L\}.$$

Theorem 3.10 tells us that

$$\lim_{n \rightarrow \infty} n^{\frac{2r}{2r+1}} \sup_{G \in \mathcal{G}_2} [R(G, \delta_n) - R(G, \delta)] \leq \frac{16B_2^2 U_a}{L} + 2B_a,$$

which is a result obtained by Karunamuni(1999). But we do not require in \mathcal{G}_2 that b_G falls in $[0, \rho_0]$, where $\rho_0 < \theta_0$ is a known constant.

§ 4 Proofs of Lemmas

Proof of Lemma 3.1 From their definitions, $\alpha_G(x)$ and $\psi_G(x)$ are monotone functions. Thus $\alpha_G(x) < \infty$ and $\psi_G(x) < \infty$ for $x > 0$ by (1.4) and (1.5). Since $\alpha_G^{(r)}(0) < B$ from (1.11), $\alpha_G(0) < \infty$, and $\psi_G(0) \leq \int_0^1 \frac{dG(\theta)}{A(\theta)} + \int_1^\infty \frac{\theta}{A(\theta)} dG(\theta) \leq \alpha_G(0) + \psi_G(1) < \infty$. So we have $\alpha_G(x) < \infty$ and $\psi_G(x) < \infty$ for all $x \geq 0$. Note that

$$\begin{aligned} \int_x^\infty \alpha_G(s) ds &= \int_0^\infty I_{[s>x]} \int_0^\infty I_{[\theta>s]} \frac{dG(\theta)}{A(\theta)} ds \\ &= \int_0^\infty I_{[\theta>x]} (\theta - x) \frac{dG(\theta)}{A(\theta)} \\ &= \int_x^\infty (\theta - x) \frac{dG(\theta)}{A(\theta)} \\ &= \psi_G(x) - x\alpha_G(x). \end{aligned}$$

Then

$$W(x) = (\theta_0 - x)\alpha_G(x) - \int_x^\infty \alpha_G(s) ds. \quad (4.1)$$

In Cases 1, 2, 4. Using Taylor's Theorem, (2.1) and (2.2), a straight-forward computation shows that

$$\begin{aligned} &E\left[\frac{K_0\left(\frac{X_j-x}{u}\right)}{ua(X_j)}\right] \\ &= \int_0^\infty \int_0^\infty \frac{K_0\left(\frac{y-x}{u}\right)}{ua(y)} \frac{a(y)}{A(\theta)} I_{[0<y<\theta]} dy dG(\theta) \\ &= \int_0^\infty \int_0^\infty \frac{1}{u} K_0\left(\frac{y-x}{u}\right) \frac{1}{A(\theta)} I_{[0<y<\theta]} dy dG(\theta) \end{aligned} \quad (4.2)$$

$$\begin{aligned}
&= \int_0^\infty \frac{1}{u} K_0\left(\frac{y-x}{u}\right) \alpha_G(y) dy \\
&= \int_0^1 K_0(t) \alpha_G(x+ut) dt \\
&= \int_0^1 K_0(t) \alpha_G(x) dt + \int_0^1 K_0(t) u t \alpha_G^{(1)}(x) dt + \cdots + \int_0^1 K_0(t) \frac{u^r t^r}{r!} \alpha_G^{(r)}(x+ut_1^*) dt \\
&= \alpha_G(x) + u^r \times \frac{1}{r!} \int_0^1 K_0(t) t^r \alpha_G^{(r)}(x+ut_1^*) dt,
\end{aligned}$$

where $0 \leq t_1^* \leq 1$. Also,

$$\begin{aligned}
&E\left[\frac{K_1\left(\frac{X_j-x}{u}\right)}{a(X_j)}\right] \tag{4.3} \\
&= \int_0^\infty \int_0^\infty \frac{K_1\left(\frac{y-x}{u}\right)}{a(y)} \frac{a(y)}{A(\theta)} I_{[0 < y < \theta]} dy dG(\theta) \\
&= \int_0^\infty \int_0^\infty K_1\left(\frac{y-x}{u}\right) \frac{1}{A(\theta)} I_{[0 < y < \theta]} dy dG(\theta) \\
&= \int_0^\infty \int_0^\infty K_1\left(\frac{y-x}{u}\right) \alpha_G(y) dy \\
&= -\left[K_1\left(\frac{y-x}{u}\right) \int_y^\infty \alpha_G(s) ds\right]_{y=x}^\infty + \frac{1}{u} \int_0^\infty \left[\int_y^\infty \alpha_G(s) ds\right] K_0\left(\frac{y-x}{u}\right) dy \\
&= \int_0^\infty K_0(t) \left[\int_{x+ut}^\infty \alpha_G(s) ds\right] dt \\
&= \int_0^\infty \left[\int_x^\infty \alpha_G(s) ds\right] dt + \int_0^\infty K_0(t) \alpha_G(t) u t dt + \cdots \\
&\quad + \int_0^\infty K_0(t) \frac{u^{r+1} t^{r+1}}{(r+1)!} \alpha_G^{(r)}(x+ut_2^*) dt \\
&= \int_x^\infty \alpha_G(s) ds + u^r \times \frac{u t^{r+1}}{(r+1)!} \int_0^\infty K_0(t) \alpha_G^{(r)}(x+ut_2^*) dt,
\end{aligned}$$

where $0 \leq t_2^* \leq 1$. Let

$$\begin{aligned}
W(x, n) &= \frac{(\theta_0 - x)}{r!} \int_0^1 t^{r+1} K_0(t) \alpha_G^{(r)}(x+ut_1^*) dt \\
&\quad - \frac{u}{(r+1)!} \int_0^1 t^{r+1} K_0(t) \alpha_G^{(r)}(x+ut_2^*) dt.
\end{aligned} \tag{4.4}$$

By (3.2), (4.1), (4.3) and (4.4), we have

$$\bar{W}(x, n) = W(x) + u^r W(x, n).$$

Since $\alpha_G^{(r)}(x) < B$ for $x \in [0, \theta_0+1]$, $x+ut_1^* \leq \theta_0+1$ and $x+ut_2^* \leq \theta_0+1$ for $x \in [d_n \wedge b_0, \theta_0]$,

$$|W(x, n)| \leq \frac{2\theta_0}{(r+1)!} B B_1 + \frac{B B_1}{(r+2)!} \equiv B_2,$$

for $x \in [d_n \wedge b_0, \theta_0]$. For Cases 3, 5, the similar computation shows that

$$\overline{W}(x, n) = W(x) + u^r W(x, n),$$

and $|W(x, n)| \leq B_2$ for $x \in [d_n \wedge b_0, \theta_0]$.

Proof of Lemma 3.2 By (1.10), $\psi_G(0) \leq \theta_0 \alpha_G(0) \leq \theta_0 B$. Thus $\psi_G(x) \leq \theta_0 B$ for $x \in [0, \theta_0]$. Then we have

$$\begin{aligned} |W(x)| &= |\theta_0 \alpha_G(x) - \psi_G(x)| \\ &\leq \theta_0 \alpha_G(x) + \psi_G(x) \\ &\leq \theta_0 \alpha_G(0) + \psi_G(0) \\ &\leq 2\theta_0 B. \end{aligned}$$

Proof of Lemma 3.3 Obviously, Z_{jn} are i.i.d. for fixed n . A few computations show that, for Cases 1, 2, 4,

$$\begin{aligned} \sigma_n^2 &= EZ_{jn}^2 \\ &\leq E[V(X_j, x, n)]^2 \\ &= \int_0^\infty \int_0^\infty \left[\frac{\theta_0 - x}{u} \frac{K_0(\frac{y-x}{u})}{a(y)} - \frac{K_1(\frac{y-x}{u})}{a(y)} \right]^2 \frac{a(y)}{A(\theta)} I_{[0 < y < \theta]} dy dG(\theta) \\ &= \int_0^\infty \frac{1}{u^2 a(y)} [(\theta_0 - x) K_0(\frac{y-x}{u}) - u K_1(\frac{y-x}{u})]^2 \alpha_G(x) dy \\ &= \int_0^1 \frac{1}{ua(x+ut)} [(\theta_0 - x) K_0(t) - u K_1(t)]^2 \alpha_G(x+ut) dt \\ &\leq \frac{B}{ua^*(x)} (2\theta_0 B_1 + B_1)^2. \end{aligned}$$

Then (3.6) is proved. And also,

$$\begin{aligned} |Z_{jn}| &= \max_{y \geq 0} \left| \frac{\theta_0 - x}{u} \frac{K_0(\frac{y-x}{u})}{a(y)} - \frac{K_1(\frac{y-x}{u})}{a(y)} \right| + |\overline{W}(x, n)| \\ &= \max_{0 \leq t \leq 1} \frac{1}{ua(x+ut)} [(\theta_0 - x) K_0(t) - u K_1(t)] + |W(x)| + B_2 \\ &\leq \frac{2\theta_0 B_1 + 1}{ua^*(x)} + 2\theta_0 B + B_2. \end{aligned}$$

For Cases 3, 5, the proof is similar. We omit it here.

Proof of Lemma 3.4 From Lemma 3.1, $|W(x, n)| \leq B_2$ for all $x \in [d_n \wedge b_0, \theta_0]$. If $W(x) > 2B_2 u^r$,

$$u^{-r} \overline{W}(x, n) = u^{-r} [W(x) + u^r W(x, n)] = u^{-r} W(x) + W(x, n) > 2B_2 - B_2 = B_2 > 0$$

and

$$\begin{aligned}\frac{W(x)}{W(x, n)} &= \frac{u^{-r}W(x)}{u^{-r}W(x) + u^rW(x, n)} \\ &\leq \frac{u^{-r}W(x) - 2B_2 + 2B_2}{u^{-r}W(x) - 2B_2 + B_2} \\ &\leq 2.\end{aligned}$$

Then (3.14) is proved. (3.15) can be proved in a similar way.

Proof of Lemma 3.5 From (1.11), we know $G'(x) = g(x)$ exists. Then

$$W'(x) = \frac{g(x)}{A(x)}(x - \theta_0).$$

Thus we have $W'(x) \leq 0$ for $x \leq \theta_0$. By (1.12), we know $W'(b_G) < 0$. Since $W'(x)$ is continuous by (1.11), we can choose c_{G1} and c_{G2} such that $c_{G1} < b_G < c_{G2} < \theta_0$ and for all $x \in [c_{G1}, c_{G2}]$,

$$-W'(x) \geq \frac{-W'(b_G)}{2} \equiv B_{G1}^{-1}.$$

On the other side, let $B_{G2} = \min\{W(c_{G1}), |W(c_{G2})|\}$. We have

$$|W(x)| \geq B_{G2}.$$

Proof of Lemma 3.6 Since $u^r \rightarrow 0$ as $n \rightarrow \infty$. There exists an integer $N_1(> N_0)$ such that for all $n > N_1$,

$$2B_2u^r < B_{G2}.$$

Then $|W(x)| \leq 2B_2u^r$ implies $x \in [c_{G1}, c_{G2}]$. So, using change of variable $y = \frac{W(x)}{2B_2u^r}$, we have

$$\begin{aligned}\int_0^{\theta_0} I_{[|W(x)| \leq 2B_2u^r]} a(x) dx &= \int_{c_{G1}}^{c_{G2}} I_{[|W(x)| \leq 2B_2u^r]} a(x) dx \\ &\leq U_G B_{G1} \int_{c_{G1}}^{c_{G2}} I_{[|W(x)| \leq 2B_2u^r]} [-W'(x)] dx \\ &\leq 2U_G B_{G1} B_2 u^r \int_{\frac{W(c_{G2})}{2B_2u^r}}^{\frac{W(c_{G1})}{2B_2u^r}} I_{[|y| \leq 1]} dy \\ &\leq 4B_2 B_{G1} U_G u^r.\end{aligned}$$

(3.11) is proved. As for (3.12), we take $N_2(> N_1)$ such that $u < \frac{c_{G1}}{2} \wedge \frac{\theta_0 - c_{G2}}{2}$ and $\eta_n \leq \min\{a(x) : 0 \leq x \leq \theta_0\}$ for cases 1, 3, 5. For $x > d_n$, by our definition of d_n , $a^*(x) \geq \eta_n$ for Cases 2, 4 and $a^*(x) \geq \min\{a(x) : 0 \leq x \leq \theta_0\} \geq \eta_n$ for Cases 1, 3, 5. For $x \in [c_{G1}, c_{G2}]$, we have $\frac{c_{G1}}{2} < x - u < x + u < c_{G2} + \frac{\theta_0 - c_{G2}}{2}$. Thus $a^*(x) \geq L_G$.

Proof of Lemma 3.7 (3.17) is from the definition of \mathcal{G}_2 . We prove (3.18) here. If $b_G - \xi > 0$, for $x \in [0, b_G - \xi]$, by the mean value theorem

$$W(b_G - \xi) - W(b_G) = -W'(x^*)\xi,$$

where $x^* \in [b_G - \xi, b_G]$. Note that $W(b_G) = 0$ and $|W'(x)| \geq L$. Thus $|W(b_G - \xi)| \geq L\xi$.

If $b_G + \xi < \theta_0$, $W(b_G + \xi) - W(b_G) = W'(x^{**})\xi$, where $x^{**} \in [b_G, b_G + \xi]$. Thus $|W(b_G + \xi)| \geq L\xi$. So for $x \in [0, 0 \vee (b_G - \xi)] \cup [(b_G + \xi) \wedge \theta_0, \theta_0]$, $|W(x)| \geq L\xi$.

Proof of Lemma 3.8

Since $u^r \rightarrow 0$ as $n \rightarrow \infty$. There exists an integer \bar{N}_1 such that for all $n > \bar{N}_1$,

$$2B_2u^r < L\xi.$$

Then $|W(x)| \leq 2B_2u^r$ implies $x \in [0 \vee (b_G - \xi), (b_G + \xi) \wedge \theta_0]$. So, using change of variable $y = \frac{W(x)}{2B_2u^r}$, we have

$$\begin{aligned} \int_0^{\theta_0} I_{|W(x)| \leq 2B_2u^r} a(x) dx &= \int_{0 \vee (b_G - \xi)}^{(b_G + \xi) \wedge \theta_0} I_{|W(x)| \leq 2B_2u^r} a(x) dx \\ &\leq U_a L^{-1} \int_{0 \vee (b_G - \xi)}^{(b_G + \xi) \wedge \theta_0} I_{|W(x)| \leq 2B_2u^r} [-W'(x)] dx \\ &\leq \frac{2U_a B_2 u^r}{L} \int_{\frac{W((b_G + \xi) \wedge \theta_0)}{2B_2u^r}}^{\frac{W(0 \vee (b_G - \xi))}{2B_2u^r}} I_{|y| \leq 1} dy \\ &\leq \frac{4B_2 U_a}{L} u^r. \end{aligned}$$

We prove (3.20) for different cases. For Case 1, $b_0 = 0$, $a^*(x) \geq L_a$. For Cases 2, 3, 4, 5, we can find $\bar{N}_2 (> \bar{N}_1)$ such that $u < \frac{b_0}{2} \wedge 1$. For $x \in [b_0, \theta_0]$, $\frac{b_0}{2} < x - u < x + u < \theta_0 + 1$. Thus $a^*(x) \geq L_a$.

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